

Recent advances on similarity solutions arising during free convection

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Abstract

This paper reviews results about free convection near a vertical flat plate embedded in some saturated porous medium. We focus on a third order autonomous differential equation that gives a special class of solutions called similarity solutions. Two cases are under consideration: in the first one we prescribe the temperature on the plate and in the second one we prescribe the heat flux on it. We will also see that the same equation appears in other industrial processes.

1 Introduction

Free convection boundary layer flows near a vertical flat plate embedded in some porous medium are studied for many years and a natural way to describe the convective flow is to look for similarity solutions. We consider two different sets of boundary conditions for the temperature on the plate: either we prescribe the temperature or we prescribe the heat flux. Both cases are leading to the same following third order non-linear autonomous differential equation

$$f''' + \alpha f f'' - \beta f'^2 = 0 \quad (1.1)$$

with the boundary conditions

$$f(0) = -\gamma, \quad f'(\infty) = 0 \text{ and } f'(0) = 1, \quad (1.2)$$

or

$$f(0) = -\gamma, \quad f'(\infty) = 0 \text{ and } f''(0) = -1. \quad (1.3)$$

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The first set of boundary conditions (1.2) with $\alpha = \frac{m+1}{2}$ and $\beta = m$ for $m \in \mathbb{R}$ corresponds to prescribed heat on the plate as in [4], [5], [12], [14], [16], [21] and [24]. The second set of boundary conditions (1.3) with $\alpha = m + 2$ and $\beta = 2m + 1$ for $m \in \mathbb{R}$ is for the prescribed surface heat flux as done in [10] and [13]. In both cases the solutions depend on two parameters: m , the power-law exponent and γ , the mass transfer parameter. For $\gamma = 0$ we have an impermeable wall, $\gamma < 0$ corresponds to a fluid suction, and $\gamma > 0$ to a fluid injection.

Equation (1.1) with suitable boundary conditions also arises in other industrial processes such as boundary layer flow adjacent to stretching walls (see [1], [2], [15], [20], [22]) or excitation of liquid metals in a high-frequency magnetic field (see [25]).

2 The case of prescribed heat

2.1 Derivation of the model

We consider a vertical permeable flat plate embedded in a porous medium at the ambient temperature T_∞ and a rectangular Cartesian co-ordinate system with the origin fixed at the leading edge of the vertical plate, the x -axis directed upward along the plate and the y -axis normal to it. If we suppose that the porous medium is homogeneous and isotropic, that all the properties of the fluid and the porous medium are constants, that the fluid is incompressible and follows the Darcy-Boussinesq law and that the temperature along the plate is varying as x^m the governing equations are

$$\begin{aligned}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\ u &= -\frac{k}{\mu} \left(\frac{\partial p}{\partial x} + \rho g \right), \\ v &= -\frac{k}{\mu} \frac{\partial p}{\partial y}, \\ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} &= \lambda \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right), \\ \rho &= \rho_\infty (1 - \beta(T - T_\infty))\end{aligned}$$

where u and v are the Darcy velocities in the x and y directions, ρ , μ and β are the density, viscosity and thermal expansion coefficient of the fluid, k is the permeability of the saturated porous medium and λ its thermal diffusivity, p is the pressure, T the temperature and g the acceleration of the gravity. The subscript ∞ is used for a value taken far from the plate. In our system of co-ordinates the boundary conditions along the plate are

$$v(x, 0) = \omega x^{\frac{m-1}{2}}, \quad T(x, 0) = T_w(x) = T_\infty + Ax^m, \quad m \in \mathbb{R},$$

with $A > 0$ and $\omega \in \mathbb{R}$ ($\omega < 0$ corresponds to a fluid suction, $\omega = 0$ is for an impermeable wall and $\omega > 0$ corresponds to a fluid injection). The boundary conditions far from the plate are

$$u(x, \infty) = 0, \quad T(x, \infty) = T_\infty.$$

If we introduce the stream function ψ such that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

and assuming that convection takes place in a thin layer around the heating plate, we obtain the boundary layer approximation

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{\rho_\infty \beta g k}{\mu} \frac{\partial T}{\partial y}, \quad (2.1)$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{1}{\lambda} \left(\frac{\partial T}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial T}{\partial y} \frac{\partial \psi}{\partial x} \right) \quad (2.2)$$

with

$$\frac{\partial \psi}{\partial x}(x, 0) = -\omega x^{\frac{m-1}{2}} \text{ and } \frac{\partial \psi}{\partial y}(x, \infty) = 0.$$

Let us introduce the new dimensionless similarity variables

$$t = (Ra_x)^{\frac{1}{2}} \frac{y}{x}, \quad \psi(x, y) = \lambda (Ra_x)^{\frac{1}{2}} f(t), \quad \theta(t) = \frac{T(x, y) - T_\infty}{T_w(x) - T_\infty}$$

with $Ra_x = (\rho_\infty \beta g k (T_w(x) - T_\infty) x) / (\mu \lambda)$ the local Rayleigh number. In terms of these variables equations (2.1) and (2.2) become

$$f'' - \theta' = 0, \quad (2.3)$$

and

$$\theta'' + \frac{m+1}{2} f \theta' - m f' \theta = 0,$$

with the boundary conditions

$$f(0) = -\gamma, \quad \theta(0) = 1,$$

and

$$f'(\infty) = 0, \quad \theta(\infty) = 0, \quad (2.4)$$

where the prime denotes differentiation with respect to t and

$$\gamma = \frac{2\omega}{m+1} \sqrt{\frac{\mu}{\rho_\infty \beta g k A \lambda}}.$$

Integrating (2.3) and taking into account the boundary conditions (2.4) leads to

$$f' = \theta$$

and the problem (1.1)-(1.2) with $\alpha = \frac{m+1}{2}$ and $\beta = m$ for $m \in \mathbb{R}$ follows.

2.2 Usefull tools

2.2.1 The initial value problem

Let $P_{m,\gamma,\alpha}$ be the following initial value problem

$$\begin{cases} f''' + \frac{m+1}{2}ff'' - mf'^2 = 0 \\ f(0) = -\gamma, \\ f'(0) = 1, \\ f''(0) = \alpha. \end{cases} \quad (2.5)$$

This first approach used is a shooting method that consists in finding values of $f''(0) = \alpha$ for which f exists on $[0, \infty)$ and such that $f'(\infty) = 0$. This direct method allows us to consider vanishing solutions but does fail in some cases (see [8]).

2.2.2 The blowing-up co-ordinates

Let us notice that if f is a solution of (1.1) then for all $\kappa > 0$ the function $t \mapsto \kappa f(\kappa t)$ is a solution too. Then, considering an interval I on which a solution f of (1.1) does not vanish, for $\tau \in I$ we can introduce the following blowing-up co-ordinates

$$\forall t \in I, \quad s = \int_{\tau}^t f(\xi) d\xi, \quad u(s) = \frac{f'(t)}{f(t)^2} \quad \text{and} \quad v(s) = \frac{f''(t)}{f(t)^3}. \quad (2.6)$$

Then, we easily get

$$\begin{cases} \dot{u} = P(u, v) := v - 2u^2, \\ \dot{v} = Q_m(u, v) := -\frac{m+1}{2}v + mu^2 - 3uv, \end{cases} \quad (2.7)$$

where the dot is for differentiating with respect to the variable s . To come back to the original problem it is sufficient to consider the initial value problem $P_{m,\gamma,\alpha}$ with $\gamma \neq 0$ and look at the trajectories of the corresponding plane dynamical system (2.7). For details, see [11].

2.3 Main results

The problem (1.1)-(1.2) appears in engineering and physical litterature, in very different context, in the middle of the previous century.

Rigorous mathematical results arise around the sixties. In [26] (Appendix 2) it is mentionned that for $\gamma = 0$ a simple explicit solution can be obtained in both cases $m = 1$ and $m = -\frac{1}{3}$ (see also [15], [21], [4] and [5]). On the other hand, the author notes that Mr J. Watson has given a simple proof that (1.1)-(1.2) has no solution for $\gamma = 0$ and $m \leq -1$. An explicit solution is also given for $m = 1$ and any γ , first in [20], and later in [22] and [8]. In these latter papers one can also find the explicit solution for $m = -\frac{1}{3}$ and any γ . Nonexistence for $\gamma = 0$ and $m = -\frac{1}{2}$ was noted in [1]. In [21], it is shown that for $\gamma = 0$ and $m < -\frac{1}{2}$ there are no solutions satisfying $f'f^2 \rightarrow 0$ at infinity.

Recently, further mathematical results concerning existence, nonexistence, uniqueness, nonuniqueness and asymptotic behaviour, are obtained in [4], [5] for $\gamma = 0$, and in [18], [8], [11], [19] and [9] for the general case.

Numerical investigations can be found in [1], [6], [12], [14], [21], [22] and [27]. In view of all these papers, the following conclusions can be drawn.

- For $m < -1$, there exists $\gamma_* > 0$ such that problem (1.1)-(1.2) has infinitely many solutions if $\gamma > \gamma_*$, one and only one solution if $\gamma = \gamma_*$, and no solution if $\gamma < \gamma_*$. For $\gamma = \gamma_*$ we have that $f(t) \rightarrow \lambda < 0$ as $t \rightarrow \infty$, and for every $\gamma > \gamma_*$ there are two solutions f such that $f(t) \rightarrow \lambda < 0$ as $t \rightarrow \infty$ and all the other solutions verify $f(t) \rightarrow 0$ as $t \rightarrow \infty$. Moreover, if f is a solution to (1.1)-(1.2), then f is negative, strictly increasing and either concave or convex-concave. (See Fig. 1 for the two solutions such that $f(t) \rightarrow \lambda < 0$ as $t \rightarrow \infty$ and three other solutions in the case $m = -2$ and $\gamma = 5$.)

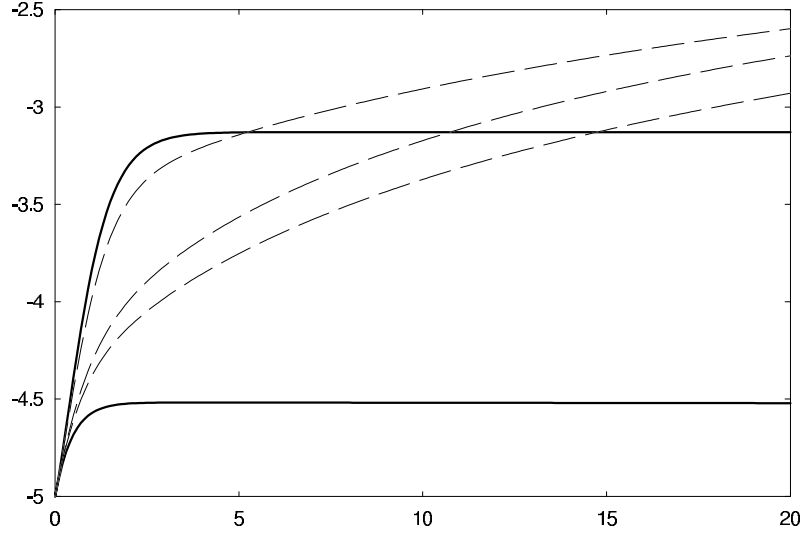


Fig. 1

- For $m = -1$ and for every $\gamma \in \mathbb{R}$, the problem (1.1)-(1.2) has no solution.
- For $-1 < m \leq -\frac{1}{2}$ and for every $\gamma \geq 0$, the problem (1.1)-(1.2) has no solution.
- For $-1 < m < -\frac{1}{2}$, there exists $\gamma_* < 0$ such that problem (1.1)-(1.2) has no solution for $\gamma_* < \gamma < 0$, one and only one solution which is bounded for $\gamma = \gamma_*$, and two bounded solutions and infinitely many unbounded solutions for $\gamma < \gamma_*$. These solutions are strictly increasing and either concave or convex-concave. (See Fig. 2 to see the two bounded solutions and four unbounded solutions in the case $m = -0.75$ and $\gamma = -10$.)
- For $-\frac{1}{2} \leq m < -\frac{1}{3}$ and for every $\gamma < 0$, the problem (1.1)-(1.2) has one bounded solution and infinitely many unbounded solutions. All these solutions are strictly increasing and either concave or convex-concave.
- For $-\frac{1}{3} \leq m < 0$ and for every $\gamma \in \mathbb{R}$, the problem (1.1)-(1.2) has an infinite number of solutions. Moreover, if $\gamma \leq 0$ one and only one solution is bounded, and if $\gamma > 0$ at least one is bounded, many infinitely are unbounded. All solutions are strictly

increasing and either concave or convex-concave. If $\gamma > 0$, the solutions becomes positive for large t .

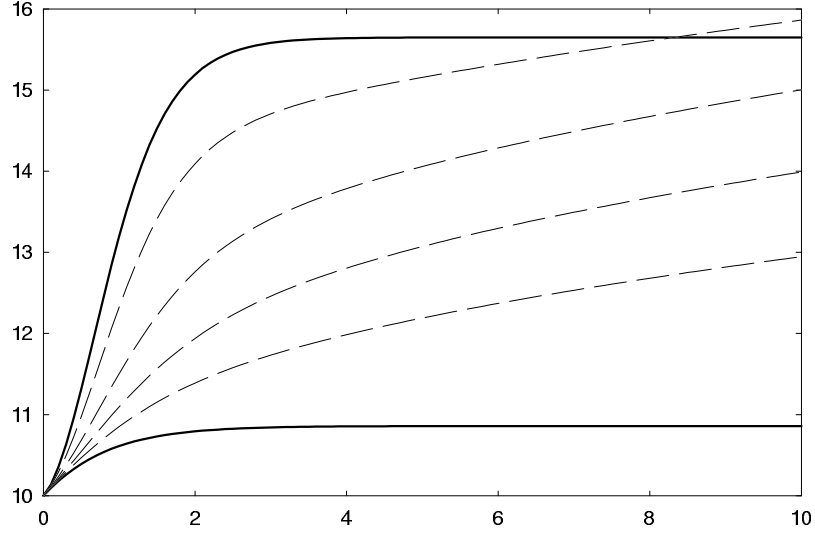


Fig. 2

- For $m \in [0, 1]$ and for every $\gamma \in \mathbb{R}$, the problem (1.1)-(1.2) has one and only one solution, moreover this solution is concave and bounded. (See Fig. 3 for the unique solution in the case $m = 0.5$ and $\gamma = 0$.)

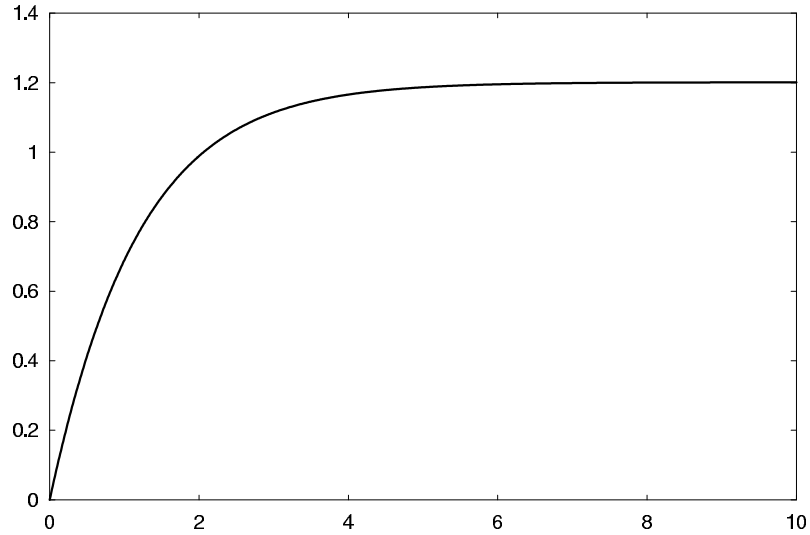


Fig. 3

- For $m > 1$ and for every $\gamma \in \mathbb{R}$, the problem (1.1)-(1.2) has one and only one concave solution and an infinite number of concave-convex solutions. All these solutions are bounded. Moreover, there is an unique concave-convex solution that verifies $f(t) \rightarrow \lambda > 0$ as $t \rightarrow \infty$ and all the other concave-convex solutions are such that $f(t) \rightarrow 0$ as $t \rightarrow \infty$. (See Fig. 4 for the unique concave solution and three

concave-convex solutions in the case $m = 1.1$ and $\gamma = 0$.)

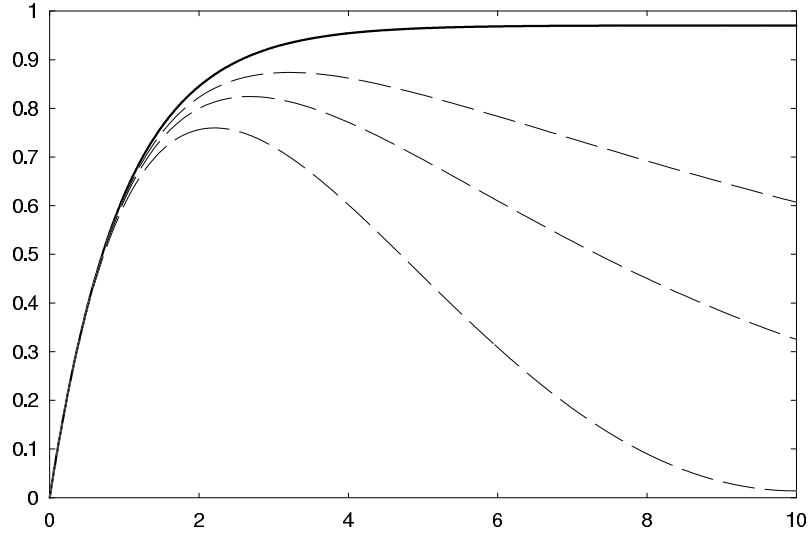


Fig. 4

Remark 2.1 The case $m = 0$ leads to the well know Blasius equation (see [3], [7]) that also is a special case of the Falkner-Skan equation (see [17]).

Remark 2.2 In [23] the authors gives results about a slightly different problem for $m = -1$ that involves pseudo-similarity.

We see from these results, that the unsolved questions concern the case $\gamma \geq 0$. More precisely, it should be interesting to try to answer to the following points

- For $-\frac{1}{2} < m < -\frac{1}{3}$, what happens for $\gamma \geq 0$?
- For $-\frac{1}{3} \leq m < 0$ and $\gamma > 0$, is there one or more bounded solutions ?

Another purpose is to compute the critical values γ_* appearing in the results above.

3 The case of prescribed heat flux

We now suppose that the plate is subjected to a variable heat flux varying as x^m and a mass transfer rate varying as $x^{\frac{m-1}{3}}$ following [13] to obtain the problem (1.1)-(1.3) with $\alpha = m + 2$ and $\beta = 2m + 1$. The mathematical study is made in [10] and leads to the following results

- For $m < -2$ there exists $\gamma_* > \sqrt[3]{\frac{2}{(m+2)^2}}$ such that the problem (1.1)-(1.3) has no solution for $\gamma < \gamma_*$, one and only one solution for $\gamma = \gamma_*$ and infinitely many solutions for $\gamma > \gamma_*$. For $\gamma = \gamma_*$ we have that $f(t) \rightarrow \lambda < 0$ as $t \rightarrow \infty$, and for every $\gamma > \gamma_*$ there are two solutions f such that $f(t) \rightarrow \lambda < 0$ as $t \rightarrow \infty$ and all the other solutions verify $f(t) \rightarrow 0$ as $t \rightarrow \infty$. Moreover, if f is a solution of (1.1)-(1.3), then f is negative, strictly concave and increasing.

- For $m = -2$ and for every $\gamma \in \mathbb{R}$, the problem (1.1)-(1.3) has no solution.
- For $-2 < m < -1$, there exists $\gamma_* < 0$ such that the problem (1.1)-(1.3) has no solution for $\gamma > \gamma_*$, one and only one solution which is bounded for $\gamma = \gamma_*$ and two bounded solutions and infinitely many unbounded solutions for $\gamma < \gamma_*$. Moreover, if f is a solution of (1.1)-(1.3), then f is positive, strictly concave, increasing and $f'(0) \geq -\frac{1}{(m+2)\gamma}$.
- For $m = -1$ the problem (1.1)-(1.3) only admits solutions for $\gamma < 0$. In this case there is an unique bounded solution with $f'(0) = -\frac{1}{\gamma}$ and an infinite number of unbounded solutions with $f'(0) > -\frac{1}{\gamma}$. Moreover all the solutions are positive, strictly concave and increasing.
- For $-1 < m < -\frac{1}{2}$ the problem (1.1)-(1.3) admits at least one bounded solution for $\gamma \in \mathbb{R}$ and many infinitely unbounded solutions for $\gamma < 0$. All these solutions are increasing and strictly concave and uniqueness of the bounded solution holds for $\gamma \leq 0$.
- For $m \geq -\frac{1}{2}$ all the solutions are bounded.
- For $-\frac{1}{2} \leq m \leq 1$ and for every $\gamma \in \mathbb{R}$ the problem (1.1)-(1.3) has one and only one solution. This solution is strictly concave and increasing. (Let us notice that for $m = -\frac{1}{2}$ we have the Blasius equation.)
- For $m > 1$ and $\gamma \in \mathbb{R}$ the problem (1.1)-(1.3) has one and only one concave solution and infinitely many concave-convex solutions. Moreover, there is an unique concave-convex solution that verifies $f(t) \rightarrow \lambda > 0$ as $t \rightarrow \infty$ and all the other concave-convex solutions are such that $f(t) \rightarrow 0$ as $t \rightarrow \infty$.

In this case it remains only two open questions

- For $-1 < m < -\frac{1}{2}$ and $\gamma > 0$, is the bounded solution unique ?
- For $-1 < m < -\frac{1}{2}$ and $\gamma \geq 0$, is there unbounded solution ?

4 Asymptotic behaviour of the unbounded solutions

For the equation (1.1) we have the following asymptotic equivalent found in [9] and [19] that holds for unbounded solutions

Theorem 4.1 *Let f be an unbounded solution of (1.1)-(1.2) or (1.1)-(1.3). There exists a constant $c > 0$ such that*

$$|f(t)| \sim ct^{\frac{\alpha}{\alpha-\beta}} \quad \text{as } t \rightarrow \infty.$$

References

- [1] W. H. H. Banks, Similarity solutions of the boundary layer equations for a stretching wall, *J. de Méchan. Théor. et Appl.* 2 (1983), pp.375-392.
- [2] W. H. H. Banks, M. B. Zaturka, Eigensolutions in boundary layer flow adjacent to a stretching wall, *IMA J. Appl. Math.* 36 (1986), pp. 263-273.
- [3] Z. Belhachmi, B. Brighi & K. Taous, On the concave solutions of the Blasius equation, *Acta Math. Univ. Comenianae*, Vol. LXIX, 2 (2000), pp.199-214.
- [4] Z. Belhachmi, B. Brighi & K. Taous, Solutions similaires pour un problème de couche limite en milieux poreux, *C. R. Mécanique* 328 (2000), pp. 407-410.
- [5] Z. Belhachmi, B. Brighi & K. Taous, On a family of differential equations for boundary layer approximations in porous media, *Euro. Jnl of Applied Mathematics*, Vol. 12, 4, Cambridge University Press (2001), pp. 513-528.
- [6] Z. Belhachmi, B. Brighi, J. M. Sac-Epee & K. Taous, Numerical simulations of free convection about a vertical flat plate embedded in a porous medium, *Computational Geosciences*, Vol. 7 (2003), pp. 137-166.
- [7] H. Blasius, Grenzsichten in Flüssigkeiten mit kleiner Reibung, *Z. Math. Phys.* 56 (1908), pp. 1-37.
- [8] B. Brighi, On a similarity boundary layer equation, *Zeitschrift für Analysis und ihre Anwendungen*, Vol. 21 (2002) 4, pp. 931-948.
- [9] B. Brighi, J.-D. Hoernel, Asymptotic behavior of the unbounded solutions of some boundary layer equation, *Archiv der Mathematik*, Vol. 85, 2 (2005), pp. 161-166.
- [10] B. Brighi, J.-D. Hoernel, On similarity solutions for boundary layer flows with prescribed heat flux. *Mathematical Methods in the Applied Sciences*, Vol. 28, 4 (2005), pp. 479-503.
- [11] B. Brighi, T. Sari, Blowing-up coordinates for a similarity boundary layer equation. *Discrete and Continuous Dynamical Systems (Serie A)*, Vol. 12 (2005) 5, pp. 929-948.
- [12] M. A. Chaudhary, J.H. Merkin & I. Pop, Similarity solutions in free convection boundary-layer flows adjacent to vertical permeable surfaces in porous media: I prescribed surface temperature, *Eur. J. Mech. B-Fluids*, 14 (1995), pp. 217-237.
- [13] M. A. Chaudhary, J.H. Merkin & I. Pop, Similarity solutions in free convection boundary-layer flows adjacent to vertical permeable surfaces in porous media: II prescribed surface heat flux, *Heat and Mass Transfer* 30, Springer-Verlag (1995), pp. 341-347.
- [14] P. Cheng, W. J. Minkowycz, Free-convection about a vertical flat plate embedded in a porous medium with application to heat transfer from a dike, *J. Geophys. Res.* 82 (14) (1977), pp. 2040-2044.

- [15] L. E. Crane, Flow past a stretching plane, *Z. Angew. Math. Phys.* 21 (1970), pp. 645-647.
- [16] E. I. Ene, D. Poliřevski, Thermal flow in porous media, D. Reidel Publishing Company, Dordrecht, 1987.
- [17] V. M. Falkner, S. W. Skan, Solutions of the boundary layer equations, *Phil. Mag.*, 7/12 (1931), pp. 865-896.
- [18] M. Guedda, Nonuniqueness of solutions to differential equations for boundary layer approximations in porous media, *C. R. Mécanique*, 330 (2002), pp. 279-283.
- [19] M. Guedda, Similarity solutions of differential equations for boundary layer approximations in porous media. To appear in *ZAMP*.
- [20] P. S. Gupta, A. S. Gupta, Heat and mass transfer on a stretching sheet with suction or blowing, *Can. J. Chem. Eng.* 55 (1977), pp. 744-746.
- [21] D. B. Ingham, S. N. Brown, Flow past a suddenly heated vertical plate in a porous medium, *J. Proc. R. Soc. Lond. A* 403 (1986), pp. 51-80.
- [22] E. Magyari, B. Keller, Exact solutions for self-similar boundary-layer flows induced by permeable stretching wall. *Eur. J. Mech. B-Fluids* 19 (2000), pp. 109-122.
- [23] E. Magyari, I. Pop, B. Keller, The 'missing' self-similar free convection boundary-layer flow over a vertical permeable surface in a porous medium, *Transport in Porous Media* 46 (2002), pp. 91-102.
- [24] J. H. Merkin, G. Zhang, On the similarity solutions for free convection in a saturated porous medium adjacent to impermeable horizontal surfaces, *Wärme und Stoffübertr.*, 25 (1990), pp. 179-184.
- [25] H. K. Moffatt, High-frequency excitation of liquid metal systems, *IUTAM Symposium: Metallurgical Application of Magnetohydrodynamics*, (1982) Cambridge.
- [26] J. T. Stuart, Double boundary layers in oscillatory viscous flow, *J. Fluid. Mech.* 24 (1966), pp. 673-687.
- [27] R. A. Wooding, Convection in a saturated porous medium at large Rayleigh number or Peclet number, *J. Fluid. Mech.*, 15 (1963), pp. 527-544.